

# BKM superalgebras from counting dyons in $\mathcal{N} = 4$ supersymmetric type II compactifications

SURESH GOVINDARAJAN<sup>\*</sup>

Department of Physics  
Indian Institute of Technology Madras  
Chennai 600036, INDIA

DILEEP P. JATKAR<sup>†</sup>

Harish-Chandra Research Institute  
Chhatnag Road, Jhusi, Allahabad 211019, INDIA

K. GOPALA KRISHNA<sup>‡</sup>

Max-Planck-Institut für Mathematik,  
Vivatsgasse 7  
53111 Bonn, Germany

**Abstract:** We study the degeneracy of quarter BPS dyons in  $\mathcal{N} = 4$  type II compactifications of string theory. We find that the genus-two Siegel modular forms generating the degeneracies of the quarter BPS dyons in the type II theories can be expressed in terms of the genus-two Siegel modular forms generating the degeneracies of quarter BPS dyons in the CHL theories and the heterotic string. This helps us in understanding the algebra structure underlying the degeneracy of the quarter BPS states. The Conway group,  $Co_1$ , plays a role similar to Mathieu group,  $M_{24}$ , in the CHL models with eta quotients appearing in the place of eta products. We construct BKM Lie superalgebra structures for the  $\mathbb{Z}_N$  (for  $N = 2, 3, 4$ ) orbifolds of the type II string compactified on a six-torus.

---

<sup>\*</sup>suresh@physics.iitm.ac.in

<sup>†</sup>dileep@hri.res.in

<sup>‡</sup>krishna@mpim-bonn.mpg.de

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Dyons in type II orbifolds</b>	<b>2</b>
2.1	The model	2
2.2	$\mathbb{Z}_N$ action from the NS5-brane	3
2.3	$\mathbb{Z}_N$ action from the Poincaré polynomial	4
2.4	Tracking dyons through dualities	5
<b>3</b>	<b>Dyon degeneracy from modular forms</b>	<b>6</b>
3.1	Counting electrically charged $\frac{1}{2}$ -BPS states	7
3.2	$\eta$ -quotients, Frame-shapes and the Conway group	9
3.3	The Siegel modular forms	10
3.4	Product Representation	12
<b>4</b>	<b>BKM superalgebras in Type II Orbifolds</b>	<b>14</b>
<b>5</b>	<b>Conclusion</b>	<b>18</b>
<b>1</b>	<b>Introduction</b>	

There has been a lot of progress in understanding quarter BPS black hole entropy in four-dimensional  $\mathcal{N} = 4$  supersymmetric string theories[1–3]. This understanding has been greatly aided by the possibility of having an exact counting of the quarter BPS dyons in these string theories [4, 5]. In recent times this has been applied to a large class of four-dimensional  $\mathcal{N} = 4$  supersymmetric theories to obtain similar degeneracy formulae for the quarter BPS dyons in these theories. In all these cases it can be written in terms of a three-dimensional contour integral of a genus-two Siegel modular form [1, 4–14]. The weight of these modular forms depend on the model at hand. In particular, the weight depends on the number of vector multiplets in the four-dimensional  $\mathcal{N} = 4$  theory. This dyon degeneracy formula, in fact, counts an index and therefore is weakly dependent on compactification moduli.

The weak moduli dependence of the degeneracy formula manifests itself in the form of walls of marginal stability across which certain dyons cease to be stable and hence do not contribute to the index[15–19]. The structure of walls of marginal stability was understood in the axion-dilaton plane for a variety of these models, which include toroidally compact-

ified heterotic string theory,  $\mathbb{Z}_N$  CHL models and  $\mathbb{Z}_N$ -orbifolds of toroidally compactified type II string theory.

Recently, it was shown that in the toroidally compactified heterotic string theory as well as in  $\mathbb{Z}_2$ ,  $\mathbb{Z}_3$ ,  $\mathbb{Z}_4$  and  $\mathbb{Z}_5$  CHL models, the dyon degeneracy formula can be written as the square of the denominator formula for some generalized Borcherds-Kač-Moody algebra. For each of the above models two families of algebras, denoted  $\mathcal{G}_N$  and  $\tilde{\mathcal{G}}_N$ , were found. The structure of walls of marginal stability was identified with the walls of Weyl chambers of the corresponding Weyl groups of these Borcherds-Kač-Moody algebras[20–24].

In this paper we would like to focus our attention on the four dimensional models with  $\mathcal{N} = 4$  supersymmetry that are obtained as asymmetric orbifolds of type II string theory on  $T^6$ . We shall refer to these models as ‘type II orbifolds’. This reflects the fact that the chain of dualities that take one from the type IIB string to the heterotic string in the CHL model takes one to the type IIA string in these examples. The type II  $\mathbb{Z}_N$ -orbifolds (for  $N = 2, 3, 4$ ) have several features in common with the CHL models. In particular, the structure of walls of marginal stability is identical to the corresponding  $\mathbb{Z}_N$  CHL models. However, the weights of the modular form are different. It is therefore natural to ask which generalized Borcherds-Kač-Moody algebra encodes the dyon degeneracy formula of type II models.

The paper is organized as follows. In section two, we provide the background for the type II orbifolds of interest as well as the relevant details of dyons in these models. In section three, we explore the modular forms that generate the degeneracies of  $\frac{1}{2}$  and  $\frac{1}{4}$  BPS states. We extend our considerations to include the  $\mathbb{Z}_4$ -orbifold and show that in all cases, we are able to express the modular forms in terms of modular forms that have already appeared in the CHL/heterotic string. The generating function of  $\frac{1}{2}$ -BPS states are  $\eta$ -quotients associated with the Conway group  $Co_1$ . In section 4, we explore the possibility of an algebraic structure with these Siegel modular forms. While we have a clear understanding of the structure for twisted dyons (i.e. dyons invariant under some symmetry) in the type II string, we have only a primitive and incomplete understanding for the one counting dyons in type II orbifolds. We conclude in section 5 with some remarks.

## 2 Dyons in type II orbifolds

### 2.1 The model

Type II string theory compactified on a six-torus has  $\mathcal{N} = 8$  supersymmetry in four dimensions. We will consider fixed-point free  $\mathbb{Z}_N$ -orbifolds ( $N = 2, 3, 4$ ) of the six-torus that preserve  $\mathcal{N} = 4$  supersymmetry. The orbifold procedure involves splitting  $T^6 = T^4 \times S^1 \times \tilde{S}^1$

and choosing the action of  $\mathbb{Z}_N$  such that it has fixed points on  $T^4$ , but this action is accompanied by a simultaneous  $1/N$  shift along the circle  $S^1$ . The total action of the orbifold is free, *i.e.*, it has no fixed points. It thus suffices to specify the action on  $T^4$ .

As we will be moving between several descriptions of the orbifold related by duality, we will need to specify the duality frame. *Description one* corresponds to type IIA string theory on a six-torus with the following  $\mathbb{Z}_N$  action. Let  $\omega = \exp(2\pi i/N)$  and  $(z_1, z_2)$  be complex coordinates on  $T^4$ . The  $\mathbb{Z}_N$  action is generated by  $(z_1, z_2) \rightarrow (\omega z_1, \omega^{-1} z_2)$ . Our considerations generalize the  $N = 2, 3$  orbifolds considered in [11].

## 2.2 $\mathbb{Z}_N$ action from the NS5-brane

The type II orbifolds of interest were studied originally by Sen and Vafa who constructed dual pairs of type II orbifolds related by U-duality [25]. In six-dimensional string-string duality, the dual string is a soliton obtained by wrapping the NS5-brane on  $T^4$ . We use this observation to obtain the Sen and Vafa result by translating the  $\mathbb{Z}_N$  action on the fields in the worldvolume of an NS5-brane wrapping  $T^4$ . Recall the fields consist of five scalars, a second-rank antisymmetric tensor (with self-dual field strength) in the bosonic sector and four chiral fermions. These are the components of a single  $(2, 0)$  tensor multiplet in  $5 + 1$  dimensions. We will dimensionally reduce the fields on  $T^4$  to obtain the fields on an effective  $1 + 1$ -dimensional theory. Using string-string duality, this theory will be that of a type IIA Green-Schwarz string in the light-cone gauge [26, 27].

Let us organise the fields in terms of  $SO(4) \times SO(4)_R$  where the first  $SO(4) = SU(2)_L \times SU(2)_R$  is from the  $T^4$  and the R-symmetry can be taken to be rotations about the four transverse directions to the NS5-brane.

1. Four scalars,  $x^m$ , are in the representation  $(1, 4_v)$ . These become four non-chiral scalars upon dimensional reduction on the four-torus.
2. The fifth scalar and the two-form antisymmetric gauge field can be combined and written as  $Y_{\alpha\beta}$  and  $Y_{\dot{\alpha}\dot{\beta}}$  where  $\alpha, \beta$  are  $SU(2)_L$  spin-half indices and  $\dot{\alpha}, \dot{\beta}$  are  $SU(2)_R$  spin-half indices. On dimensional reduction on the four-torus, the  $Y_{\alpha\beta}$  become the four left-moving chiral bosons and the  $Y_{\dot{\alpha}\dot{\beta}}$  become four right-moving chiral bosons. When combined with the four non-chiral bosons, they become the Green-Schwarz bosons in the light-cone gauge of the type IIA string.
3. The fermions are  $\psi_{A\beta}$  and  $\psi_{A\dot{\beta}}$  where  $A$  is a spinor index of  $SO(4)_R$ . These become the left- and right-moving fermions in the effective  $1 + 1$ -dimensional theory — these are the Green-Schwarz fermions in the light-cone gauge of the type IIA string.

In the above set up, it is easy to work out transformations of fields under  $\mathbb{Z}_N$  subgroup of  $SU(2)_L$ . The group element belonging to the  $\mathbb{Z}_N$  subgroup of  $SU(2)_L$  is given by

$$g_\alpha^\beta \equiv \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad (2.1)$$

where  $\omega = \exp(2\pi i/N)$  for  $N = 2, 3, 4$ .

One can see that the only fields that transform under this action are those that carry the index  $\alpha$ . Thus, we see that the chiral fermions all transform as

$$\psi_{A\alpha} \rightarrow g_\alpha^\beta \psi_{A\beta}. \quad (2.2)$$

Thus, we see that four of the fermions pick up the phase  $\omega$  and the other four pick up the phase  $\omega^{-1}$ . The field  $Y_{\alpha\beta}$  transforms as

$$Y_{\alpha\beta} \rightarrow g_\alpha^\gamma g_\beta^\delta Y_{\gamma\delta}. \quad (2.3)$$

Thus, two fields are invariant under the  $\mathbb{Z}_N$  and the other two transform with phases  $\omega^2$  and  $\omega^{-2}$ . All other fields are invariant under the  $\mathbb{Z}_N$  action.

In the dimensional reduction of the the (2,0) theory on  $T^4$ , the  $SU(2)_L$  fields get mapped to (say) left-movers and the  $SU(2)_R$  fields get mapped to right-movers. Thus, we see that the orbifold has a chiral action. In particular, the four bosons that arise from  $Y_{\alpha\beta}$  give rise to four *left-moving chiral bosons* and the  $\psi_{A\alpha}$  give rise to four *left-moving chiral fermions*.

### 2.3 $\mathbb{Z}_N$ action from the Poincaré polynomial

Consider the Poincaré polynomial for  $T^4$ . Recall that two of the one-forms pick up a phase  $\omega$  while the other two pick up a phase  $\omega^{-1}$  under the  $\mathbb{Z}_N$  action. We incorporate this into the Poincaré polynomial and obtain the action on all harmonic forms on  $T^4$ .

$$(1 - \omega x)^2 (1 - \omega^{-1} x)^2 = x^4 - 2x^3\omega - \frac{2x^3}{\omega} + x^2\omega^2 + \frac{x^2}{\omega^2} + 4x^2 - 2x\omega - \frac{2x}{\omega} + 1. \quad (2.4)$$

In the above expansion, we identify even powers of  $x$  with bosons in the 1 + 1 dimensional theory and odd powers with fermions. The coefficient of a term in the polynomial gives the orbifold action on the corresponding field. Thus six of the bosons are always periodic and the other two have fractional moding determined by the phase.

We now present the details of the orbifold action for the Green-Schwarz superstring that we just derived.

[N=2]  $\omega = -1$  implies that  $\omega^2 = 1$ . Thus, one has eight periodic bosons and eight anti-periodic fermions.

[N=3]  $\omega = \exp(2\pi i/3)$ . One has six periodic bosons and two bosons with periodicity three. Four fermions go to  $\omega$  times themselves and the other four go to  $\omega^{-1}$  times themselves.

[N=4]  $\omega = i$ . One has six periodic bosons and two anti-periodic bosons. Four fermions go to  $\omega$  times themselves and the other four go to  $\omega^{-1}$  times themselves.

Thus, the second description gives rise to an asymmetric orbifold of the type IIA string on  $T^6$  and thus is analogous to CHL compactifications of the heterotic string. In the CHL examples, recall that the heterotic string arises from the type IIA NS5-brane wrapping  $K3$  in the place of  $T^4$  that we considered.

## 2.4 Tracking dyons through dualities

Recall, the quarter BPS dyons possess charges which are mutually non-local and therefore they do not appear in the perturbative spectrum of the theory. The electric charge vector  $\mathbf{Q}_e$  and the magnetic charge vector  $\mathbf{Q}_m$  of a state are defined in the second description. One of the reasons for this choice is the similarity of this description to the CHL description. To see this more explicitly, we will describe a dyonic state in terms of a system containing  $Q_5$  D5 branes wrapping  $T^4 \times S^1$ ,  $Q_1$  D1 branes wrapping  $S^1$ ,  $J$  units of momentum along  $\tilde{S}^1$ ,  $-n$  units of momentum along  $S^1$  and a Kaluza-Klein monopole charged with respect to the gauge field along  $\tilde{S}^1$ .

This description is related to the second description by a chain of duality transformations. This chain involves first an S-duality transformation, which takes D-branes to NS-branes leaving all other quantum numbers unaffected. This is followed by a T-duality along the circle  $\tilde{S}^1$ . This transformation takes us from type IIB theory to type IIA theory and replaces the Kaluza-Klein monopole by a single NS5 brane wrapped on  $T^4 \times S^1$ ,  $Q_5$  NS5 branes by  $Q_5$  Kaluza-Klein monopoles along  $\hat{S}^1$  and  $J$  units of momentum along  $\tilde{S}^1$  is replaced by  $J$  fundamental strings wrapping  $\hat{S}^1$ , where  $\hat{S}^1$  is a circle T-dual to  $\tilde{S}^1$ . The  $\mathbb{Z}_N$ -orbifold action involves  $\mathbb{Z}_N$ -orbifold of  $T^4$  and simultaneous  $1/N$  unit of shift along  $S^1$ . Since the orbifolded circle is not participating in the T-duality transformation, the orbifold action commutes with the T-duality transformation. Finally, one carries out string-string duality to arrive at the second description. The action of this string-string duality transformation is identical to the string-string duality transformation which relates type IIA theory compactified on  $K3$  to heterotic string theory compactified on  $T^4$ , namely, all fundamental strings are replaced by NS5 branes and vice versa. Thus, in the end we have  $Q_1$  Kaluza-Klein monopoles along  $\hat{S}^1$ ,  $-n$  units of momentum along  $S^1$ ,  $J$  NS5 branes wrapping  $\hat{T}^4 \times \hat{S}^1$ ,  $Q_1$  NS5 branes wrapping  $\hat{T}^4 \times S^1$ , and a single fundamental string wrapping  $S^1$ .

The second description exclusively contains description in terms of fundamental strings, NS5 branes, Kaluza-Klein monopoles and momenta. If we denote momenta along  $S^1 \times \hat{S}^1$  by  $\mathbf{n}$ , fundamental string winding charges along them by  $\mathbf{w}$  and NS5 brane, and Kaluza-Klein monopole charges by  $\mathbf{N}$  and  $\mathbf{W}$  respectively then the T-duality invariants constructed from these electric and magnetic charges are

$$\mathbf{Q}_e^2 = 2\mathbf{n} \cdot \mathbf{w}, \quad \mathbf{Q}_m^2 = 2\mathbf{N} \cdot \mathbf{W}, \quad \mathbf{Q}_e \cdot \mathbf{Q}_m = \mathbf{n} \cdot \mathbf{N} + \mathbf{w} \cdot \mathbf{W}. \quad (2.5)$$

It is easy to check that these T-duality invariants take the following values before the orbifold action,

$$\frac{1}{2}\mathbf{Q}_e^2 = n, \quad \frac{1}{2}\mathbf{Q}_m^2 = Q_1 Q_5, \quad \mathbf{Q}_e \cdot \mathbf{Q}_m = J. \quad (2.6)$$

The  $\mathbb{Z}_N$ -orbifold action commutes with the entire duality chain and is therefore well defined in any duality frame. It is convenient for us to discuss it in the second description so that we can easily read out its effect on dyonic charges. The  $\mathbb{Z}_N$ -orbifold acts by  $1/N$  shift along  $S^1$ , which results in reducing the radius of the circle by a factor of  $N$ . Thus, the fundamental unit of momentum along  $S^1$  is  $N$  and hence momentum along  $S^1$  in the orbifolded theory becomes  $n/N$ . To maintain  $J$  NS5 branes transverse to  $S^1$  after the orbifold we need to start with  $N$  copies of  $J$  NS5 branes symmetrically arranged on  $S^1$  before orbifold. The resulting configuration has

$$\boxed{\frac{1}{2}\mathbf{Q}_e^2 = n/N, \quad \frac{1}{2}\mathbf{Q}_m^2 = Q_1 Q_5, \quad \mathbf{Q}_e \cdot \mathbf{Q}_m = J,} \quad (2.7)$$

in the orbifolded theory.

The S-duality symmetry of this theory in the second description is related to the T-duality symmetry in the original type IIB description. The  $1/N$  shift along  $S^1$  breaks the S-duality symmetry of the second description to  $\Gamma_1(N)$ .

### 3 Dyon degeneracy from modular forms

As mentioned in the previous section, computing the dyon spectrum is non-trivial because dyons do not appear in the perturbative spectrum of string theory. In fact, dyon counting necessarily requires computing the degrees of freedom coming from the solitonic sector of the theory. The dyon degeneracy formula can be obtained in two different ways, giving rise to either the additive formula or the multiplicative one.

As shown in [5], there are two modular forms that one constructs – one is the generating function of the dyon degeneracies denoted by  $\tilde{\Phi}_k(\mathbf{Z})$  and another, denoted by  $\Phi_k(\mathbf{Z})$ , is the generating function of twisted dyons in the CHL string.<sup>1</sup> Let us call the corresponding

---

<sup>1</sup>One has  $\mathbf{Z} = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \in \mathcal{H}_2$  where  $\mathcal{H}_2$  is the Siegel upper-half space.

modular forms in the type II orbifolds to be  $\tilde{\Psi}_k(\mathbf{Z})$  and  $\Psi_k(\mathbf{Z})$ . The weight  $k$  of the Siegel modular form is given by

$$k + 2 = \frac{12}{N + 1} ,$$

when  $(N + 1) | 12$  i.e.,  $N = 2, 3$ . For  $N = 4$ , one has  $k = 1$ .

### 3.1 Counting electrically charged $\frac{1}{2}$ -BPS states

As mentioned earlier, we will define our charges in the second description. In this case, electrically charged states appear as excitations of the type IIA string. In particular, the degeneracy is dominated by the contribution from the twisted sector states. We will compute the electrically charged states in a twisted sector.  $\frac{1}{2}$ -BPS states arise when the right-movers are in the ground state and we allow all excitations that are consistent with level matching as was done for the heterotic string in ref. [28].

$N = 1$

As a warm-up, consider the left-movers of the type IIA string on  $T^6$ . In the Ramond sector and in the light-cone gauge, one has eight periodic bosons and periodic fermions. All oscillators, bosonic and fermionic, have integer moding and the Witten index is given by the product of the bosonic and fermionic contributions:

$$\mathcal{W}_B \times \mathcal{W}_F = \left( \frac{1}{\prod_n (1 - q^n)} \right)^8 \times \left( \prod_n (1 - q^n) \right)^8 = 1 . \quad (3.1)$$

This is expected as there is a perfect cancellation of bosonic and fermionic contributions in the Witten index. Of course, the oscillator partition function is not unity and equals

$$\mathcal{Z}_B \times \mathcal{Z}_F = \left( \frac{1}{\prod_n (1 - q^n)} \right)^8 \times \left( \prod_n (1 + q^n) \right)^8 = \frac{\eta(2\tau)^8}{\eta(\tau)^{16}} . \quad (3.2)$$

$N = 2$

The eight periodic bosons have integer moding and each contribute a factor of  $\eta(\tau)^{-1}$  to the Witten index while the eight anti-periodic fermions each have half-integer moding and contribute  $\eta(\tau/2)/\eta(\tau)$ . One has

$$\begin{aligned} \mathcal{W}_B \times \mathcal{W}_F &= \left( \frac{1}{\prod_n (1 - q^n)} \right)^8 \times \left( \prod_n (1 - q^{n+1/2}) \right)^8 \\ &= \frac{\eta(\tau/2)^8}{\eta(\tau)^{16}} = \frac{1}{g_{\tilde{\rho}}(\tau/2)} , \end{aligned} \quad (3.3)$$



where the Frame-shape  $\tilde{\rho} = 1^{-8}2^{16}$ .<sup>2</sup> We can also compute the twisted index for BPS states in the type II string that are invariant under the  $\mathbb{Z}_2 = (-1)^{F_L}$  action without accompanied by shift along any circle. These states contribute to the twisted helicity supertrace  $B_4^{\hat{g}}$ [30], where  $\hat{g} = (-1)^{F_L}$  is the generator under which these states are invariant. In type II theory on  $T^6$  this corresponds to setting all R-R fields and R-NS fields to zero. This means we are left with NS-NS and NS-R sector fields in the subspace of moduli space where  $(-1)^{F_L}$  symmetry is manifest. States contributing to the twisted helicity supertrace  $B_4^{\hat{g}}$  belong to the elementary string states with arbitrary excitation in the left moving sector but ground state in the right moving sector. These states break all 16 left moving supersymmetries and 8 right moving supersymmetries. However, only 8 right moving supersymmetries are invariant under  $(-1)^{F_L}$ , as a result these states contribute to  $B_4^{\hat{g}}$ . With some abuse of language, henceforth, we will refer to these states as twisted dyons, which are in fact 1/8-BPS states and are counted by the twisted helicity supertrace  $B_6^{\hat{g}}$ [30]. The  $\eta$ -quotient for these states is given by the S-transform i.e.,  $\tau \rightarrow -1/\tau$  of the  $\eta$ -quotient that we got from the Witten index. This leads to a second  $\eta$ -quotient (ignoring numerical factors)

$$\frac{1}{g_{\rho}(\tau)} = \frac{\eta(2\tau)^8}{\eta(\tau)^{16}}, \quad (3.4)$$

with Frame-shape  $\rho = 1^{16}2^{-8}$ . Unlike the CHL examples, where the S-transform did not change the cycle shape, we obtain *a pair of Frame-shapes* in all the examples – one that counts an index for  $\frac{1}{2}$ -BPS states in the orbifold of the type II string while the other counts a twisted index for  $\frac{1}{4}$ -BPS states in the type II string.

$N = 3$

The six periodic bosons have integer moding and each contribute a factor of  $\eta(\tau)^{-1}$  to the Witten index. While the two other bosons have fractional moding of  $\pm 1/3$ . The fermions each have fractional moding of  $\pm 1/3$  and contribute  $\eta(\tau/3)/\eta(\tau)$ . One has

$$\begin{aligned} \mathcal{W}_B \times \mathcal{W}_F &= \frac{1}{\prod_n (1 - q^n)^6 (1 - q^{n+1/3}) (1 - q^{n-1/3})} \times \prod_n (1 - q^{n+1/3})^4 (1 - q^{n-1/3})^4 \\ &= \frac{\eta(\tau/3)^3}{\eta(\tau)^9} = \frac{1}{g_{\tilde{\rho}}(\tau/3)}, \end{aligned} \quad (3.5)$$

where the Frame-shape  $\tilde{\rho} = 1^{-3}3^9$ . Counting twisted dyons leads to a second  $\eta$ -quotient

$$\frac{1}{g_{\rho}(\tau)} = \frac{\eta(3\tau)^3}{\eta(\tau)^9}, \quad (3.6)$$

with Frame-shape  $\rho = 1^93^{-3}$ .

---

<sup>2</sup>Frame-shapes are natural generalizations of cycles shapes that appear in the CHL examples. (see ref. [29]). The map  $g_{\rho}(\tau)$  that maps a Frame-shape  $\rho = 1^{a_1}2^{a_2}\dots$  to the  $\eta$ -quotient  $g_{\rho}(\tau) = \eta(\tau)^{a_1}\eta(2\tau)^{a_2}\dots$  is identical to the one that appeared in the CHL strings[23].

$N = 4$

The six periodic bosons have integer moding and each contribute a factor of  $\eta(\tau)^{-1}$  to the Witten index. While the two other bosons are anti-periodic and have half-integral moding. The fermions each have fractional moding of  $\pm 1/4$  and contribute  $\eta(\tau/4)/\eta(\tau)$ . One has

$$\begin{aligned}\mathcal{W}_B \times \mathcal{W}_F &= \frac{1}{\prod_n (1 - q^n)^6 (1 - q^{n+1/2})^2} \times \prod_n (1 - q^{n+1/4})^4 (1 - q^{n-1/4})^4 \\ &= \frac{\eta(\tau/4)^4}{\eta(\tau)^4 \eta(\tau/2)^6} = \frac{1}{g_{\tilde{\rho}}(\tau/4)},\end{aligned}\tag{3.7}$$

where the Frame-shape  $\tilde{\rho} = 1^{-4}2^64^4$ . Counting twisted dyons leads to a second  $\eta$ -quotient

$$\frac{1}{g_{\rho}(\tau)} = \frac{\eta(4\tau)^4}{\eta(\tau)^4 \eta(2\tau)^6},\tag{3.8}$$

with Frame-shape  $\rho = 1^42^64^{-4}$ .

### 3.2 $\eta$ -quotients, Frame-shapes and the Conway group

The counting of  $\frac{1}{2}$ -BPS states in the type II orbifold is given by  $\eta$ -quotients that are associated with the Frame-shapes  $\tilde{\rho}$  while twisted  $\frac{1}{4}$ -BPS states in the type II string are associated with Frame-shapes  $\rho$  as given in Table 1. This nicely generalizes the corresponding result for CHL strings where the generating functions were given by  $\eta$ -products corresponding to cycle shapes.

The appearance of the  $\eta$ -quotients and Frame-shapes may be understood as follows. It is known that the Conway group  $Co_1$  arises as the group of automorphisms of the algebra of chiral vertex operators in the NS sector of the superstring[31]. Any symmetry of finite order of the chiral superstring *must* thus be an element of  $Co_1$ . It is known that the conjugacy classes of  $Co_1$  are given by Frame-shapes. It thus appears that the Conway group plays a role analogous to the one played by the Mathieu group  $M_{24}$  with  $\eta$ -quotients replacing  $\eta$ -products[23]. Is there a moonshine for the Conway group? The  $\eta$ -quotients for  $N = 2, 3$  have also appeared in the work of Scheithauer who constructed the Fake monster superalgebra as well as noted the connection with the Conway group[32–34].

Multiplicative  $\eta$ -quotients have been systematically studied by Martin and he has provided a list of 71 such quotients – almost all appear to be associated to conjugacy classes of  $Co_1$ [35]. Table 1 is a subset of the list except for the ones for  $N = 2$ . The  $\eta$ -quotients for  $N = 2$  violate the multiplicative condition of Martin – he requires them to be eigenforms of all Hecke operators. The ones for  $N = 2$  are not eigenforms for  $T_2$  as can be easily checked<sup>3</sup>. It appears that the condition imposed by Martin might be too strong

---

<sup>3</sup>We thank Martin for useful correspondence which clarified this point.

as it excludes the  $N = 2$  quotient that we obtain. It might be sufficient to require that the  $\eta$ -quotient be a Hecke eigenform for all Hecke operators  $T_m$  with  $(m, N) = 1$ . The  $\eta$ -quotients for  $N = 2, 3$  have already been derived in [11] and our results agree with the expressions given there.

$k$	$\tilde{\rho}$	$\rho$	$\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$N$	$G$
2	$1^{-8}2^{16}$	$1^{16}2^{-8}$		2	$\mathbb{Z}_2$
1	$1^{-3}3^9$	$1^93^{-3}$	$\left(\frac{-3}{d}\right)$	3	$\mathbb{Z}_3$
1	$1^{-4}2^64^4$	$1^42^64^{-4}$	$\left(\frac{-1}{d}\right)$	4	$\mathbb{Z}_4$

**Table 1.**  $\eta$ -quotients with  $N \leq 4$ :  $\tilde{\rho}$  and  $\rho$  are the pair of Frame-shapes,  $k + 2$  is the weight of the  $\eta$ -quotient.

### 3.3 The Siegel modular forms

We will look for genus-two Siegel modular forms,  $\Psi_k(\mathbf{Z})$  and  $\tilde{\Psi}_k(\mathbf{Z})$ , that have the following behavior

$$\begin{aligned} \lim_{z \rightarrow 0} \Psi_k(\mathbf{Z}) &\sim z^2 g_{\tilde{\rho}}(\tau) g_{\tilde{\rho}}(\sigma) , \\ \lim_{z \rightarrow 0} \tilde{\Psi}_k(\mathbf{Z}) &\sim z^2 g_{\rho}(\tau/N) g_{\tilde{\rho}}(\sigma) . \end{aligned} \quad (3.9)$$

**Remark:** The fractional charges here appear with the Frame-shape  $\rho$  rather than the Frame-shape  $\tilde{\rho}$  that we saw in the half-BPS counting in the orbifold theory. This is related to the fact that the type II dyon formula can be brought into the form identical to the CHL dyon formula by doing the substitution  $\mathbf{Q}_m^2 \rightarrow 2\mathbf{Q}_e^2$  and  $\mathbf{Q}_e^2 \rightarrow \mathbf{Q}_m^2/2$  [11].

We rewrite the  $\eta$ -quotients,  $g_{\tilde{\rho}}(\tau)$ , in a suggestive manner as

$$\begin{aligned} g_{1^{-8}2^{16}}(\tau) &= \frac{\eta^{16}(2\tau)}{\eta^8(\tau)} = \frac{\eta^{16}(2\tau)\eta^{16}(\tau)}{\eta^{24}(\tau)} , \\ g_{1^{-3}3^9}(\tau) &= \frac{\eta^9(3\tau)}{\eta^3(\tau)} = \frac{\eta^9(3\tau)\eta^9(\tau)}{\eta^{12}(\tau)} , \\ g_{1^{-4}2^64^4}(\tau) &= \frac{\eta(4\tau)^4\eta(2\tau)^6}{\eta(\tau)^4} = \frac{[\eta(4\tau)^4\eta(2\tau)^2\eta(\tau)^4] \times [\eta(\tau)^4\eta(2\tau)^4]}{\eta(\tau)^{12}} . \end{aligned} \quad (3.10)$$

In this form, it is easy to see that the  $\eta$ -quotients can be written as quotients of the  $\eta$ -products (or their square-roots) that appear in the  $\frac{1}{2}$ -BPS counting of the heterotic string and the CHL orbifolds. This naturally suggests that the Siegel modular forms  $\Psi_k(\mathbf{Z})$  can be written as quotients of the Siegel modular forms that appear in the heterotic string and

its CHL orbifolds[5, 21]. Concretely, we conjecture that

$$\Psi_2^{N=2}(\mathbf{Z}) = \frac{\Phi_6(\mathbf{Z})^2}{\Phi_{10}(\mathbf{Z})}, \quad (3.11)$$

$$\Psi_1^{N=3}(\mathbf{Z}) = \frac{\Delta_2(\mathbf{Z})^3}{\Delta_5(\mathbf{Z})}, \quad (3.12)$$

$$\Psi_1^{N=4}(\mathbf{Z}) = \frac{\Phi_3(\mathbf{Z})\Delta_3(\mathbf{Z})}{\Delta_5(\mathbf{Z})}, \quad (3.13)$$

where  $\Delta_{k/2}(\mathbf{Z})^2 = \Phi_k(\mathbf{Z})$ , and  $\Phi_k(\mathbf{Z})$  are Siegel modular forms appearing in CHL models. It is easy to see that the first condition in Eq.(3.9) is easily satisfied. Further, the  $\tilde{\Psi}_k(\mathbf{Z})$  can be defined by the S-transform[5, 12, 23, 36] ( $\text{vol}_\rho \equiv \prod_{j=1}^N (j)^{a_j}$ )

$$\tilde{\Psi}_k(\mathbf{Z}) := (\text{vol}_\rho)^{1/2} \tau^{-k} \Psi_k(\tilde{\mathbf{Z}}), \quad (3.14)$$

with

$$\tilde{\tau} = -1/\tau, \quad \tilde{z} = z/\tau, \quad \tilde{\sigma} = \sigma - z^2/\tau.$$

We have arrived at the relation (3.11), (3.12), and (3.13) by comparing the additive seeds for Siegel modular forms for heterotic, CHL and type II models. We need to be careful about the relation between the corresponding Siegel modular forms because it could involve some phases(multiplier system); particularly so, if the relation involves taking square roots of the Siegel modular forms or if embedding of the genus one congruent subgroup  $\Gamma_0(N)$  in the subgroup of  $Sp(2, \mathbb{Z})$  gives rise to a multiplier system for the Siegel modular form. In the case of  $\mathbb{Z}_2$ -orbifold, the type II dyon partition function is a ratio of the square of the  $\mathbb{Z}_2$  CHL Siegel modular form and the Siegel modular form for the heterotic string theory. We are in a fortunate situation here because neither of them have any multiplier system with respect to the subgroup of  $Sp(2, \mathbb{Z})$  which includes the genus one congruent subgroup  $\Gamma_0(2)$ . It is therefore obvious that  $\Psi_2(\mathbf{Z})$  does not have a multiplier system. In case of  $\mathbb{Z}_3$  also the CHL Siegel modular form does not have a multiplier system but the type II Siegel modular form is not directly related to the Siegel modular form of the  $\mathbb{Z}_3$  CHL model or for that matter to that of the heterotic theory. It is a ratio of cube of  $\Delta_2(\mathbf{Z})$  and  $\Delta_5(\mathbf{Z})$ , where  $\Delta_2(\mathbf{Z})$  is square root of the Siegel modular form for the  $\mathbb{Z}_3$  CHL model and  $\Delta_5(\mathbf{Z})$  is the square root of the heterotic string theory Siegel modular form. Therefore  $\Psi_1^{N=3}(\mathbf{Z})$ , in general, can have multiplier system under the subgroup of  $Sp(2, \mathbb{Z})$  which includes  $\Gamma_0(3)$ . It is therefore important to check that the Siegel modular form  $\Psi_1^{N=3}$  as well as  $\Psi_1^{N=4}$ , for which genus one congruent subgroup is  $\Gamma_0(4)$ , do not possess a multiplier system. In addition to this, it is important to ensure that the Taylor expansion of the inverse powers of these Siegel modular forms, in terms of  $\rho$ ,  $\sigma$  and  $v$  have integer coefficients. While it is

desirable to do these checks we will derive these results using the product formula and this derivation will automatically ensure that both these conditions are satisfied.

### 3.4 Product Representation

The product formula is written in terms of twisted elliptic genus of  $T^4$ . For  $\mathcal{N} = 4$  type II orbifolds, the  $\mathbb{Z}_N$  acts simultaneously on a circle of  $T^2$  by a  $1/N$  shift and on  $T^4$  by  $2\pi/N$  rotation. For  $N = 2$ , it reverses all  $T^4$  coordinates whereas for  $N = 3$ , the orbifold action is easily seen if we choose lattice vectors which subtend  $2\pi/3$  angle with each other.

The product formula is given in terms of Jacobi forms of weight zero and index one,  $F_{N\ II}^{(r,s)}(\tau, z)$ , which are defined by

$$F_{N\ II}^{(r,s)}(\tau, z) \equiv \frac{1}{N} \text{Tr}_{RR; \tilde{g}^r} \left( \tilde{g}^s (-1)^{F+\bar{F}} e^{2\pi i \tau L_0} e^{2\pi i \mathcal{J} z} \right), \quad r, s = 0, 1, \dots, N-1, \quad (3.15)$$

where  $\tilde{g}$  is a transformation which implements the  $\mathbb{Z}_N$  orbifold transformation on the coordinates of  $T^4$ . While in case of the  $\mathbb{Z}_2$  orbifold it changes signs of all four coordinates, for  $\mathbb{Z}_3$  orbifold it generates a  $2\pi/3$  rotation in a two-dimensional plane and  $-2\pi/3$  rotation in the orthogonal two-dimensional plane.  $F$  and  $\bar{F}$  are left and right chiral fermions in the  $(4, 4)$  superconformal field theory with  $T^4/\mathbb{Z}_N$  target space. This superconformal field theory has  $SU(2)_L \times SU(2)_R$  R-symmetry and  $\mathcal{J}/2$  represents the generator of the  $U(1)_L$  subgroup of this R-symmetry.

The twisted elliptic genera  $F_{N\ II}^{(r,s)}(\tau, z)$  are Jacobi forms and have the following Fourier-Jacobi expansion

$$F_{N\ II}^{(r,s)}(\tau, z) = \sum_{b=0}^1 \sum_{\substack{j \in 2\mathbb{Z}_b, n \in \mathbb{Z}/N \\ 4n-j^2 \geq -b^2}} c_b^{(r,s)}(4n-j^2) e^{2\pi i n \tau + 2\pi i j z}. \quad (3.16)$$

Explicit product formulae in terms of the Fourier-Jacobi coefficients  $c_b^{(r,s)}(4n-j^2)$  for  $\Psi(\mathbf{Z})$  and  $\tilde{\Psi}(\mathbf{Z})$  as well as the CHL modular forms have been given, for instance, in ref. [12]. We do not reproduce them here as we do not need the detailed expressions for our analysis.

For  $N = 2$  and 3, the  $F^{(r,s)}(\tau, z)$  can be written as[11]

$$\begin{aligned} F_{N\ II}^{(0,0)}(\tau, z) &= 0 \\ F_{N\ II}^{(0,s)}(\tau, z) &= \frac{16}{N} \sin^4 \left( \frac{\pi s}{N} \right) \frac{\vartheta_1 \left( \tau, z + \frac{s}{N} \right) \vartheta_1 \left( \tau, -z + \frac{s}{N} \right)}{\vartheta_1 \left( \frac{s}{N} \right)^2}, \\ &\quad \text{for } 1 \leq s \leq N-1, \\ F_{N\ II}^{(r,s)}(\tau, z) &= \frac{4N}{(N-1)^2} \frac{\vartheta_1 \left( \tau, z + \frac{s}{N} + \frac{r}{N} \tau \right) \vartheta_1 \left( \tau, -z + \frac{s}{N} + \frac{r}{N} \tau \right)}{\vartheta_1 \left( \frac{s}{N} + \frac{r}{N} \tau \right)^2}, \\ &\quad \text{for } 1 \leq r \leq N-1, 0 \leq s \leq N-1. \end{aligned} \quad (3.17)$$

However, it is more instructive to write them in terms of twisted elliptic genera of the CHL models and that of the heterotic string theory. Let us first note that the twisted elliptic genera of the CHL  $\mathbb{Z}_N$ -orbifold,  $F_{N \text{ CHL}}^{(r,s)}(\tau, z)$ , can be written as[5] (for  $N = 2, 3, 5, 7$ )

$$F_{N \text{ CHL}}^{(0,0)}(\tau, z) = \frac{8}{N} A(\tau, z) \quad (3.18)$$

$$F_{N \text{ CHL}}^{(0,s)}(\tau, z) = \frac{8}{N(N+1)} A(\tau, z) - \frac{2}{N+1} B(\tau, z) E_N(\tau) \quad \text{for } 1 \leq s \leq (N-1) \quad (3.19)$$

$$F_{N \text{ CHL}}^{(r,rk)}(\tau, z) = \frac{8}{N(N+1)} A(\tau, z) + \frac{2}{N(N+1)} B(\tau, z) E_N\left(\frac{\tau+k}{N}\right) \quad \text{for } 1 \leq r \leq (N-1), 0 \leq k \leq (N-1), \quad (3.20)$$

where,

$$A(\tau, z) = \left[ \frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} \right],$$

$$B(\tau, z) = \eta(\tau)^{-6} \vartheta_1(\tau, z)^2,$$

and the Eisenstein series for the congruent subgroup  $\Gamma_1(N)$  is given by

$$E_N(\tau) = \frac{12i}{\pi(N-1)} \partial_\tau [\ln \eta(\tau) - \ln \eta(N\tau)] = 1 + \frac{24}{N-1} \sum_{\substack{n_1, n_2 \geq 1 \\ n_1 \not\equiv 0 \pmod{N}}} n_1 e^{2\pi i n_1 n_2 \tau}.$$

For the  $\mathbb{Z}_2$ -orbifold of the type II model,  $F^{(r,s)}(\tau, z)$  can be written as<sup>4</sup>

$$F_{N=2 \text{ II}}^{(r,s)}(\tau, z) = \begin{cases} 2F_{N=2 \text{ CHL}}^{(r,s)}(\tau, z) - F_{N=1 \text{ Het.}}^{(r,s)}(\tau, z), & (r, s) = (0, 0) \\ 2F_{N=2 \text{ CHL}}^{(r,s)}(\tau, z), & (r, s) \neq (0, 0). \end{cases} \quad (3.21)$$

Similarly, for the  $\mathbb{Z}_3$ -orbifold of the type II model, one has

$$F_{N=3 \text{ II}}^{(r,s)}(\tau, z) = \begin{cases} \frac{3}{2}F_{N=3 \text{ CHL}}^{(r,s)}(\tau, z) - \frac{1}{2}F_{N=1 \text{ Het.}}^{(r,s)}(\tau, z), & (r, s) = (0, 0) \\ \frac{3}{2}F_{N=3 \text{ CHL}}^{(r,s)}(\tau, z), & (r, s) \neq (0, 0). \end{cases} \quad (3.22)$$

Thus, we see that the seed for the product representation also confirms the fact that  $\Psi_2^{N=2}(\mathbf{Z})$  and  $\Psi_1^{N=3}(\mathbf{Z})$  can be written in terms of the Siegel modular forms that appear in the CHL models and the heterotic string theory, as stated in (3.11) and (3.12). Since, the elliptic genera for the type II  $\mathbb{Z}_4$ -orbifold have not been worked out, our expression for  $\Psi_1^{N=4}(\mathbf{Z})$  implies that twisted elliptic genera can be written in terms of the CHL ones as follows:

$$F_{N=4 \text{ II}}^{(0,0)}(\tau, z) = F_{N=4 \text{ CHL}}^{(0,0)}(\tau, z) + \frac{1}{2}F_{N=2 \text{ CHL}}^{(0,0)}(\tau, z) - \frac{1}{2}F_{N=1 \text{ Het.}}^{(0,0)}(\tau, z) = 0, \quad (3.23)$$

---

<sup>4</sup>This observation was already made in a footnote appearing in [11] though the implication that the type II Siegel modular form can be written in terms of the CHL Siegel modular forms was not made.

and for  $(r, s) \neq (0, 0)$ , one has

$$F_{N=4}^{(r,s)}(\tau, z) = \begin{cases} F_{N=4}^{(r,s)}{}_{CHL}(\tau, z) + \frac{1}{2}F_{N=2}^{(r,s)}{}_{CHL}(\tau, z), & (r, s) = (0, 0) \pmod{2} \\ F_{N=4}^{(r,s)}{}_{CHL}(\tau, z), & (r, s) \neq (0, 0) \pmod{2}. \end{cases} \quad (3.24)$$

While we have not proved the above formulae for the  $N = 4$  twisted elliptic genera, it is easy to see that it passes simple checks. For instance,  $F_{II}^{(0,0)}(\tau, z) = 0$  as expected. A second check is that  $\Psi_1^{N=4}(\mathbf{Z})$  is a modular form (with character) of a level 4 subgroup of  $Sp(2, \mathbb{Z})$  and is invariant under the S-duality group  $\Gamma_1(4)$  – this follows from the known behavior of the CHL modular forms[23].

#### 4 BKM superalgebras in Type II Orbifolds

In their original construction, the automorphic forms constructed by Borchers[37] via the singular theta lift also happened to be related to infinite dimensional Lie superalgebras. The automorphic form appears as the denominator identity of the BKM Lie algebra. The infinite product representation generated by the theta lift formed the product side of the denominator identity giving the set of roots of the algebra and the multiplicity of these roots, while the Fourier expansion of the automorphic form formed the sum side of the identity which gives the Weyl group and its action on the roots. This idea was also used by Scheithauer[32], who constructed the singular theta lifts for the elements of the Mathieu group  $M_{23}$  and  $Co_1$  and showed the existence of BKM Lie superalgebras for the constructed automorphic forms. The same idea was also applied to the modular forms constructed in the CHL theory, and the existence of BKM Lie superalgebras corresponding to these genus-two Siegel modular forms was shown[20, 21, 23, 24, 36].

Given that the Siegel modular forms generating the dyonic degeneracies in the type II models can be expressed in terms of ratio of the Siegel modular forms that appear in the description of the quarter BPS dyons in the heterotic and  $\mathbb{Z}_N$ -orbifolded CHL strings, the possibility arises that one can construct BKM Lie superalgebras corresponding to the type II modular forms also. We explore this possibility in this section. Before we do so, however, we remark on an important point from the corresponding constructions in the CHL theory as well as Borchers's work.

In the case of  $\mathbb{Z}_N$ -orbifolded CHL theories with  $N > 1$ , there exists more than one cusp, and the modular form has more than one product expansion corresponding to each of the cusps. Thus, interpreted as the denominator identity of an infinite dimensional Lie algebra, each of the different expansions corresponds to different BKM Lie superalgebras. The modular forms at the different cusps correspond (in some sense) to twisted and untwisted counting of dyonic degeneracies. This phenomenon is not particular to the CHL models.

An example in a context different from the CHL theory is that of the denominator identity of the fake monster superalgebra which has two completely different algebras corresponding to the cusps at level 1 and 2 (see[37] Example 13.7). Again, the algebras at the two cusps correspond to twisted and untwisted counting of states. The level 1 cusp gives a BKM Lie superalgebra for superstrings on a  $T^{10}$ , while the level 2 cusp corresponds to a twisted denominator formula corresponding to an automorphism that is 1 on the Bosonic elements and  $-1$  on the Fermionic ones. Remarkably, the two algebras are completely different from each other, with different Weyl vectors, Weyl groups (the level 2 algebra has a trivial Weyl group), and different real simple roots (the level 1 algebra has an infinite number of real simple roots, while the level 2 algebra has no real simple roots at all). Thus we see that the BKM Lie superalgebras corresponding to the expansions about the different cusps are quite different from each other.

Coming back to the type II models, we now try to find the algebraic structures, if any, occurring in the type II models. The method we will use is the general prescription outlined in[21, see Appendix D.1] for understanding the algebra structure from the expansion of the modular form. The modular forms occurring in the type II models seem less amenable to an algebraic interpretation than their CHL counterparts. Where the modular forms occurring in the CHL theory seemed to be related to two families of BKM Lie superalgebras, the modular forms in the type II theories have no such obvious forms. Nevertheless, we have guidance from the fact that the modular forms of the type II theory are related to the ones occurring in the CHL theory and this can help us derive some algebraic structure from the corresponding structures in the CHL theory. However, the extent of this insight gets limited by the fact that the modular forms of the type II theory are given as quotients of the CHL modular forms, and this does not always translate into a straightforward interpretation of the characters of representations.

For the  $N = 2$  orbifold, the modular forms at the two cusps, given in terms of the CHL modular forms are

$$\Psi_2(\mathbf{Z}) = \frac{\Phi_6(\mathbf{Z})^2}{\Phi_{10}(\mathbf{Z})}, \quad \text{and} \quad \tilde{\Psi}_2(\mathbf{Z}) = \frac{\tilde{\Phi}_6(\mathbf{Z})^2}{\Phi_{10}(\mathbf{Z})}. \quad (4.1)$$

Using the CHL examples as a guide, the modular forms relevant for the BKM Lie superalgebras are the square roots of the above modular forms and are given as

$$\Xi_1(\mathbf{Z}) = \frac{\Delta_3(\mathbf{Z})^2}{\Delta_5(\mathbf{Z})}, \quad \text{and} \quad \tilde{\Xi}_1(\mathbf{Z}) = \frac{\tilde{\Delta}_3(\mathbf{Z})^2}{\Delta_5(\mathbf{Z})}. \quad (4.2)$$

Let us start with the modular form  $\Xi_1$  to look for a BKM Lie superalgebra, if any, associated to it. Since  $\Xi_1$  is given as the ratio of  $\Delta_3$  and  $\Delta_5$ , both of which have the BKM



Lie superalgebras,  $\mathcal{G}_2$  and  $\mathcal{G}_1$  respectively, associated to them, we look at the corresponding algebras,  $\mathcal{G}_2$  and  $\mathcal{G}_1$ .

The algebras  $\mathcal{G}_1$  and  $\mathcal{G}_2$  have the same set of real simple roots as  $\mathcal{G}_1 - \alpha_1, \alpha_2$  and  $\alpha_3$  with Cartan matrix[21, 38]

$$A_{1,II} \equiv \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}. \quad (4.3)$$

The difference between the two algebras is in their imaginary roots. For instance, the multiplicity of light-like roots,  $t\eta_0$ , is given by the formula ( $\eta_0$  is a primitive light-like root and  $t \in \mathbb{Z}_{>0}$ )

$$1 - \sum_{t \in \mathbb{N}} m(t\eta_0) q^n = \prod_{n \in \mathbb{N}} (1 - q^n)^{\frac{k-4}{2}} (1 - q^{Nn})^{\frac{k+2}{2}}, \quad (4.4)$$

where  $k$  for  $\mathcal{G}_1$  and  $\mathcal{G}_2$  is 10 and 6 respectively. Now a quotient of the modular forms suggests the real simple roots of the resulting algebra will remain same, since two copies of the roots exist in the numerator  $\Delta_3^2$ , while one copy of the roots is cancelled by the presence of  $\Delta_5$ . This seems to suggest the algebra corresponding to  $\Xi_1$  will be one with the same real simple roots as  $\mathcal{G}_1$ . The multiplicity of the light-like roots can be guessed from the eq. (4.4) and using the quotient form of  $\Xi_1$  in terms of  $\Delta_3$  and  $\Delta_5$ . The multiplicity of the light-like roots,  $t\eta_0$ , for  $\mathcal{G}_{II,2}$  is given by

$$1 - \sum_{t \in \mathbb{N}} m(t\eta_0) q^n = \prod_{n \in \mathbb{N}} (1 - q^n)^{-8} (1 - q^{2n})^8. \quad (4.5)$$

Thus, we see that one can associate a BKM Lie superalgebra,  $\mathcal{G}_{II,2}$ , to the modular form  $\Xi_1$ .

Turning to the modular form at the other cusp, we see it can be written as the quotient of the modular forms  $\tilde{\Delta}_3$  and  $\Delta_5$ . We might hope to repeat the above success in obtaining  $\mathcal{G}_{II,2}$  to find a BKM Lie superalgebra associated to the modular form  $\tilde{\Xi}_1$ , but it is not so simple at the other cusp. The algebra  $\tilde{\mathcal{G}}_2$  (associated with  $\tilde{\Delta}_3$ ) has four real simple roots, of which two are different from that occurring in  $\mathcal{G}_1$ . In particular, one of the real simple roots of  $\mathcal{G}_1$  will appear as a pole in the denominator identity. It is potentially a fermionic simple root. Thus, considering only the real simple roots of BKM Lie superalgebra that might be associated with the modular form  $\tilde{\Xi}_1$ , we expect to see four bosonic (even) real simple roots (those appearing in  $\tilde{\mathcal{G}}_2$ ) and a fermionic (odd) real simple root (the one contributing to the pole). Even in the finite case, the structure of Lie superalgebras is technically more complicated than the one for classical Lie algebras. Gritsenko and Nikulin, in their analysis of BKM Lie superalgebras consider only those superalgebras *without* odd real simple roots i.e., these superalgebras only contain odd imaginary simple roots[38, see

Appendix]. The BKM Lie superalgebras that appear in the CHL dyon counting as well as those in Scheithauer's Fake Monster Lie superalgebra belong to this category[33]. We defer a detailed discussion for the future.

It is instructive to write eq.(4.1) in the following form

$$\Phi_6(\mathbf{Z}) \times \Phi_6(\mathbf{Z}) = \Psi_2(\mathbf{Z}) \times \Phi_{10}(\mathbf{Z}), \quad (4.6)$$

and similarly for  $\tilde{\Psi}_2(\mathbf{Z})$ . Written in this form we can interpret it as decomposition of product of two characters belonging to trivial representations of the BKM algebra corresponding to two copies of  $N = 2$  CHL in terms of a character of trivial representation of the BKM algebra corresponding to the heterotic theory and  $N = 2$  type II model. This is quite analogous to the way coset models were constructed using affine Kač-Moody symmetry, and in particular to the way characters of the coset models were derived. It is worth pointing out that the following sequence of Cartan matrices shows relation of the BKM algebra associated with  $\tau = i\infty$  cusp of the Heterotic string and its CHL analogs to simple and affine  $SU(2)$  Lie algebra,

$$A_1 = \begin{pmatrix} 2 \end{pmatrix} \hookrightarrow A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \hookrightarrow A_{1,II} \equiv \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}. \quad (4.7)$$

With this close analogy it is tempting to speculate that the type II models possess a BKM generalization of the affine Goddard-Kent-Olive type coset symmetry[39]. It would be interesting to explore this relation further. It will also enable us to give an unified description of type II orbifold models as BKM cosets.

We next come to the case of the  $\mathbb{Z}_3$ -orbifold. The modular forms generating the degeneracy of the  $\frac{1}{4}$ -BPS states for the  $\mathbb{Z}_3$ -orbifold can be written, in terms of the CHL modular forms as

$$\Psi_1^{N=3}(\mathbf{Z}) = \frac{\Delta_2(\mathbf{Z})^3}{\Delta_5(\mathbf{Z})}, \quad \tilde{\Psi}_1^{N=3}(\mathbf{Z}) = \frac{\tilde{\Delta}_2(\mathbf{Z})^3}{\Delta_5(\mathbf{Z})}. \quad (4.8)$$

We cannot take a square-root as we did for the  $N = 2$  case as it leads to modular forms with non-integral coefficients. Nevertheless, we observe that the real bosonic simple roots that appear in the denominator of  $\Psi_1(\mathbf{Z})$  are cancelled by one  $\Delta_2(\mathbf{Z})$ . It leaves behind two sets of three real simple roots with Cartan matrix  $A_{1,II}$  – this is the Cartan matrix of a rank three Lorentzian Lie algebra[38]. Thus, the modular form appears to be the product of at least two BKM Lie superalgebras both of which are inequivalent automorphic extensions of the Lorentzian Lie algebra with Cartan matrix  $A_{1,II}$ [38]. Physically, it means that the walls of marginal stability of twisted dyons is identical to that of the  $N = 2$  orbifold. The same conclusion is obtained for the  $N = 4$  modular form  $\Psi_1^{N=4}(\mathbf{Z})$ . However, we do not

have a complete understanding of the algebraic structure for modular forms that generate degeneracies of  $\frac{1}{4}$ -BPS in the type II orbifolds, i.e., for the  $\tilde{\Psi}_1^{N=3}(\mathbf{Z})$  and  $\tilde{\Psi}_1^{N=4}(\mathbf{Z})$  as in the  $N = 2$  example.

Thus, we see that for the type II models one can associate a BKM Lie superalgebra structure to the expansion of the modular forms generating the degeneracy of twisted  $\frac{1}{4}$ -BPS states along the same lines as for the algebras in the CHL models. However, only the expansion about one of the cusps admits an algebra structure, while it is not very clear if an algebra exists about the other cusp.

## 5 Conclusion

In this work we have attempted to understand the degeneracy of the quarter BPS dyons in  $\mathbb{Z}_N$ -orbifolds of  $\mathcal{N} = 4$  type II compactification of string theory by studying the genus-two Siegel modular forms generating the degeneracies of these states. We see that the Siegel modular forms for the  $\mathbb{Z}_N$ -orbifolds in the type II compactifications are expressible in terms of the CHL Siegel modular forms. This relation exists for both the generating functions of the half as well as the quarter BPS states of the type II theory. Using this we construct the algebraic structures underlying these degeneracies. The algebras underlying twisted dyons in the type II string are all inequivalent automorphic corrections to the rank three Lorentzian Kac-Moody algebra with Cartan matrix  $A_{1,II}$  consistent with the physical requirement that they have identical walls of marginal stability.

**Acknowledgments:** We thank Y. Martin and U. Ray for useful conversations as well as correspondence. One of us, DPJ, thanks IITM for hospitality during a visit where some of the work reported was carried out.

## References

- [1] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, *Gen. Rel. Grav.* **40** (2008) 2249–2431, [[arXiv:0708.1270](#)].
- [2] A. Sen, *Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity*, *JHEP* **09** (2005) 038, [[hep-th/0506177](#)].
- [3] G. Lopes Cardoso, B. de Wit, J. Kappeli, and T. Mohaupt, *Asymptotic degeneracy of dyonic  $\mathcal{N} = 4$  string states and black hole entropy*, *JHEP* **12** (2004) 075, [[hep-th/0412287](#)].
- [4] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, *Counting Dyons in  $\mathcal{N} = 4$  String Theory*, *Nucl. Phys.* **B484** (1997) 543–561, [[hep-th/9607026](#)].
- [5] D. P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018, [[hep-th/0510147](#)].

- [6] D. Shih, A. Strominger, and X. Yin, *Recounting dyons in  $\mathcal{N} = 4$  string theory*, *JHEP* **10** (2006) 087, [[hep-th/0505094](#)].
- [7] D. Gaiotto, *Re-counting dyons in  $\mathcal{N} = 4$  string theory*, [hep-th/0506249](#).
- [8] J. R. David, D. P. Jatkar, and A. Sen, *Product representation of dyon partition function in CHL models*, *JHEP* **06** (2006) 064, [[hep-th/0602254](#)].
- [9] A. Dabholkar and S. Nampuri, *Spectrum of Dyons and Black Holes in CHL orbifolds using Borchers Lift*, *JHEP* **11** (2007) 077, [[hep-th/0603066](#)].
- [10] J. R. David and A. Sen, *CHL dyons and statistical entropy function from D1-D5 system*, *JHEP* **11** (2006) 072, [[hep-th/0605210](#)].
- [11] J. R. David, D. P. Jatkar, and A. Sen, *Dyon spectrum in  $\mathcal{N} = 4$  supersymmetric type II string theories*, *JHEP* **11** (2006) 073, [[hep-th/0607155](#)].
- [12] J. R. David, D. P. Jatkar, and A. Sen, *Dyon spectrum in generic  $\mathcal{N} = 4$  supersymmetric  $\mathbb{Z}_N$  orbifolds*, *JHEP* **01** (2007) 016, [[hep-th/0609109](#)].
- [13] A. Dabholkar and D. Gaiotto, *Spectrum of CHL dyons from genus-two partition function*, *JHEP* **12** (2007) 087, [[hep-th/0612011](#)].
- [14] N. Banerjee, D. P. Jatkar, and A. Sen, *Adding charges to  $N = 4$  dyons*, *JHEP* **07** (2007) 024, [[arXiv:0705.1433](#)].
- [15] A. Sen, *Walls of Marginal Stability and Dyon Spectrum in  $\mathcal{N} = 4$  Supersymmetric String Theories*, *JHEP* **05** (2007) 039, [[hep-th/0702141](#)].
- [16] A. Dabholkar, D. Gaiotto, and S. Nampuri, *Comments on the spectrum of CHL dyons*, *JHEP* **01** (2008) 023, [[hep-th/0702150](#)].
- [17] M. C. N. Cheng and E. Verlinde, *Dying Dyons Don't Count*, *JHEP* **09** (2007) 070, [[arXiv:0706.2363](#)].
- [18] S. Banerjee, A. Sen, and Y. K. Srivastava, *Genus Two Surface and Quarter BPS Dyons: The Contour Prescription*, *JHEP* **03** (2009) 151, [[arXiv:0808.1746](#)].
- [19] A. Sen, *Two Centered Black Holes and  $\mathcal{N} = 4$  Dyon Spectrum*, *JHEP* **09** (2007) 045, [[arXiv:0705.3874](#)].
- [20] M. C. Cheng and A. Dabholkar, *Borchers-Kac-Moody Symmetry of  $\mathcal{N} = 4$  Dyons*, *Commun.Num.Theor.Phys.* **3** (2009) 59–110, [[arXiv:0809.4258](#)].
- [21] S. Govindarajan and K. Gopala Krishna, *Generalized Kac-Moody Algebras from CHL dyons*, *JHEP* **04** (2009) 032, [[0807.4451](#)].
- [22] M. C. Cheng and E. P. Verlinde, *Wall Crossing, Discrete Attractor Flow, and Borchers Algebra*, *SIGMA* **4** (2008) 068, [[arXiv:0806.2337](#)].
- [23] S. Govindarajan and K. Gopala Krishna, *BKM Lie superalgebras from dyon spectra in  $\mathbb{Z}_N$  CHL orbifolds for composite  $N$* , *JHEP* **1005** (2010) 014, [[arXiv:0907.1410](#)].
- [24] K. Gopala Krishna, *BKM Lie superalgebra for the  $\mathbb{Z}_5$ -orbifolded CHL string*, [hep-th/1011.2168](#).
- [25] A. Sen and C. Vafa, *Dual pairs of type II string compactification*, *Nucl. Phys.* **B455** (1995) 165–187, [[hep-th/9508064](#)].
- [26] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, *BPS spectrum of the five-brane and black hole entropy*, *Nucl. Phys.* **B486** (1997) 77–88, [[hep-th/9603126](#)].

- [27] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, *BPS quantization of the five-brane*, *Nucl. Phys.* **B486** (1997) 89–113, [[hep-th/9604055](#)].
- [28] A. Sen, *Black holes and the spectrum of half-BPS states in  $\mathcal{N} = 4$  supersymmetric string theory*, *Adv. Theor. Math. Phys.* **9** (2005) 527–558, [[hep-th/0504005](#)].
- [29] G. Mason, *Frame-shapes and rational characters of finite groups*, *J. Algebra* **89** (1984), no. 2 237–246.
- [30] A. Sen, *A Twist in the Dyon Partition Function*, *JHEP* **1005** (2010) 028, [[arXiv:0911.1563](#)].
- [31] J. F. Duncan, *Super-moonshine for Conway’s largest sporadic group*, *Duke Math. J.* **139** (2007), no. 2 255–315.
- [32] N. R. Scheithauer, *Generalized Kac-Moody algebras, automorphic forms and Conway’s group. I*, *Adv. Math.* **183** (2004), no. 2 240–270.
- [33] N. R. Scheithauer, *The fake monster superalgebra*, *Adv. Math.* **151** (2000), no. 2 226–269.
- [34] N. R. Scheithauer, *Generalized Kac-Moody algebras, automorphic forms and Conway’s group. II*, *J. Reine Angew. Math.* **625** (2008) 125–154.
- [35] Y. Martin, *Multiplicative  $\eta$ -quotients*, *Trans. Amer. Math. Soc.* **348** (1996), no. 12 4825–4856.
- [36] S. Govindarajan, *BKM Lie superalgebras from counting twisted CHL dyons*, *JHEP* **1105** (2011) 089, [[arXiv:1006.3472](#)].
- [37] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, *Invent. Math.* **132** (1998), no. 3 491–562.
- [38] V. A. Gritsenko and V. V. Nikulin, *Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras*, *Amer. J. Math.* **119** (1997), no. 1 181–224, [[alg-geom/9504006](#)].
- [39] P. Goddard, A. Kent, and D. Olive, *Unitary representation of the Virasoro and super-Virasoro algebras*, *Comm. Math. Phys.* **103** (1986), no. 1 105–119.